Probabilistic behaviour of induced order statistics from bivariate rayleigh distribution

Sunil Kumar[1], Jagmohan Singh Dhakar[2,*]

[1]Assistant Professor, Dept. of Statistics, RBS College, Agra, Uttar Pradesh, [2]Dept. of Community Medicine, Santosh Medical College, Ghaziabad, Uttar Pradesh

***Corresponding Author:**
Email: jagmohansinghdhakar@gmail.com

Abstract

In the present paper a new bivariate distribution has been derived from univariate Rayleigh distribution using the concept given by Mongestern (1956). This bivariate Rayleigh distribution is then considered to obtain the Induced (concomitants) order statistics.

In the section 1 characterization of univariate Rayleigh distribution and bivariate Rayleigh distribution has been provided. In section 2 and 3 the density function of order statistics and induced order statistics have been obtained. To characterize the model the moments of induced order statistics have been also obtained. The moment generating functions and cummulant generating function has been obtained. The joint distribution of two concomitants is also obtained in section 4.

Keywords: Rayleigh Distribution, Induced Order Statistics

Characteristics of Rayleigh distribution & bivariate

- **Rayleigh Distribution:** In probability theory and Statistics, the Rayleigh distribution is a continuous probability distribution. A Rayleigh distribution is often observed when the overall magnitude of a vector is related to its directional components. The distribution can also arise in the case of random complex numbers whose real and imaginary components are i.i.d. Gaussian. In that case, the absolute value of the complex number is Rayleigh–distributed. The distribution is name after Lord Rayleigh.

  The p.d.f of Rayleigh distribution is

$$f(x) = \frac{xe^{-x^2/2\sigma^2}}{\sigma^2} \qquad 0 \le x \le \infty \qquad \ldots(1)$$

The cumulative distribution function of above distribution is

$$F(x) = \int_o^x f(x)dx \qquad 0 \le x \le \infty$$

$$= 1 - e^{-x^2/2\sigma^2}$$

The reliability function at time t say R (T) = P (T>t) is

$$R(X) = p(X > x) = e^{-x^2/2^{\sigma 2}}$$

The arithmetic mean for the Rayleigh distribution is

$$E(X) = \int_0^\infty xf(x)dx = \sigma\sqrt{\frac{\pi}{2}}$$

The variance for the Rayleigh distribution is

$$V(X) = E(X^2) - (E(X))^2 = \frac{4-\pi}{2}\sigma^2$$

The median of the Rayleigh distribution is

$$Me = \sigma\sqrt{\log(4)}$$

Bivariate Rayleigh Distribution: A bivariate distribution can be developed by using the concept of Mongestern (1956) as follows:

Let X and Y follows univariate Rayleigh distribution with p.d.f. defined in eq. (1)

Now, to derive a bivarite distribution the following form is used.

$$F(x,y) = F(x)F(y)\left[1+\delta(1-F(x)\right]\left[1-F(y)\right] \qquad \ldots(2)$$

Let put δ=-1

$$=1-e^{-x^2/2\sigma^2}-e^{-y^2/2\sigma^2}+e^{\frac{-x^2-2y^2}{2\sigma^2}}+e^{\frac{-2x^2-y^2}{2\sigma^2}}-e^{\frac{-x^2-y^2}{\sigma^2}} \quad \ldots(3)$$

Probability density function: The p.d.f. of the bivariate Rayleigh distribution will be

$$f(x,y)=\frac{d^2}{dx\,dy}\left[f(x,y)\right]=\frac{2xy}{\sigma^4}\left[e^{\frac{-x^2-2y^2}{2\sigma^2}}+e^{\frac{-2x^2-y^2}{2\sigma^2}}-2e^{\frac{-x^2-y^2}{\sigma^2}}\right] \quad \ldots(4)$$

It can be proved that $\iint f(x,y)\,dx\,dy=1$

The conditional pdf of x given y is

$$f(x/y)=\frac{f(x,y)}{f(y)}=\frac{2x}{\sigma^2}\left[e^{\frac{-x^2-y^2}{2\sigma^2}}+e^{\frac{-2x^2}{2\sigma^2}}-2e^{\frac{-2x^2-y^2}{2\sigma^2}}\right] \quad \ldots(5)$$

and conditional pdf of y given x is

$$f(y/x)=\frac{f(x,y)}{f(x)}=\frac{2y}{\sigma^2}\left[e^{\frac{-2y^2}{2\sigma^2}}+e^{\frac{-x^2-y^2}{2\sigma^2}}-2e^{\frac{-x^2-2y^2}{2\sigma^2}}\right] \quad \ldots(6)$$

- **Probability density function of induced order statistics:** In this section, density function of Order Statistics and their Induced(Concomitants) have been obtained. For Rayleigh distribution the pdf of the r^th order statistics can be obtained as:

$$f_{r:n}=\frac{n!}{(r-1)!(n-r)!}\left[F(x)^{r-1}(1-F(x)\right]^{n-r} f(x)$$

$$=C_{r:n}\ \frac{x}{\sigma^2}\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)^{r-1}\left(e^{\frac{-x^2}{2\sigma^2}}\right)^{n-r+1} \quad \ldots(7)$$

For r=1 the pdf of the first order statistics

$$=C_{1:n}\ \frac{x}{\sigma^2}\left(e^{\frac{-x^2}{2\sigma^2}}\right)^{n} \quad \ldots(8)$$

For r=n the probability density function of the n^th order statistics

$$=C_{n:n}\ \frac{x}{\sigma^2}\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)^{n-1} e^{\frac{-x^2}{2\sigma^2}} \quad \ldots(9)$$

The probability density function of the n^th concomitant of the order statistic for r=n is,

$$g_{(n:n)}(y)=\int f(y/x)\, f_{n:n}(x)\,dx$$

$$=\int_0^\infty \frac{2y}{\sigma^2}\left(e^{\frac{-2y^2}{2\sigma^2}}+e^{\frac{-x^2-y^2}{2\sigma^2}}-2e^{\frac{-x^2-2y^2}{2\sigma^2}}\right) C_{n:n}\ \frac{x}{\sigma^2}\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)^{n-1} e^{\frac{-x^2}{2\sigma^2}}\,dx \quad \ldots(A)$$

This expression can be solved as

$$g_{(n:n)}(y)= =\frac{2y}{\sigma^2}\ e^{\frac{-y^2}{\sigma^2}}+\frac{2yn}{\sigma^2 B(2,n)}e^{\frac{-y^2}{2\sigma^2}}-\frac{4ny}{\sigma^2 B(2,n)}\ e^{\frac{-y^2}{\sigma^2}} \quad \ldots(10)$$

It can be proved that $\int g_{(n:n)}(y)\,dy=1$.

The probability density function of $y_{(r:n)}$ i.e. r^{th} Induced order statistic will be

$$g_{(r:n)}(y) = \sum_{i=r}^{n} (-1)^{i-r}\ i-1_{C_r}\ n_{C_i}\ g_{(i:i)}(y)$$

$$= \sum_{i=r}^{n} (-1)^{i-r}\ i-1_{C_r}\ n_{C_i} \left[\frac{2y}{\sigma^2} e^{-y^2/\sigma^2} + \frac{2yi}{\sigma^2 B(2,i)} e^{-y^2/2\sigma^2} - \frac{4i}{\sigma^2 B(2,i)} y\, e^{-y^2/\sigma^2} \right] \quad \ldots(11)$$

We can obtain the probability density function of first concomitant of the order statistics by putting r=1 in (11) as follows:

$$g_{(1:n)}(y) = \sum_{i=r}^{n} (-1)^{i-r}\ i-1_{C_r} \left[\frac{2y}{\sigma^2} e^{-y^2/\sigma^2} + \frac{2y}{\sigma^2 B(2,1)} e^{-y^2/2\sigma^2} - \frac{4}{\sigma^2 B(2,1)} y\, e^{-y^2/\sigma^2} \right] \quad \ldots(12)$$

Moments of $y_{(n:n)}$:

Now we will derive the expression for k^{th} moment of $y_{(n:n)}$ k=0,1,2,…….n.

$$\mu^k_{(n:n)} = E\left[y^k_{(n:n)} \right] = \int_0^\infty y^k\ g_{(n:n)}(y)\, dy$$

$$= \int_0^\infty y^k \left[\frac{2y}{\sigma^2} e^{-y^2/\sigma^2} + \frac{2yn}{\sigma^2 B(2,n)} e^{-y^2/2\sigma^2} - \frac{4n}{\sigma^2 B(2,n)} y\, e^{-y^2/\sigma^2} \right] dy \quad \ldots(B)$$

This expression for moment can be solved as

$$= \sigma^k \overline{\lfloor k/2+1} \left[1 + \frac{2n\sigma^k 2^{k/2+1}}{B(2,n)} - \frac{2n}{B(2,n)} \right] \quad \ldots(13)$$

Similarly one can obtain the expression for k^{th} moments of $y_{(r:n)}$ as:

$$\mu^k_{(r:n)} = E\left[y^k_{(n:n)} \right] = \int_0^\infty y^k\ g_{(n:n)}(y)\, dy$$

$$= \sum_{i=r}^{i} (-1)^{i-r}\ i-1_{C_{r-1}} \left[\sigma^k \overline{\lfloor k/2+1} \left[1 + \frac{i\, 2^{k/2+1}}{B(2,i)} - \frac{2i}{B(2,i)} \right] \right. \quad \ldots(14)$$

3. Moment generating function and Cumulant generating function of $y_{[n:n]}$:

Moment generating function of $y_{(n:n)}$ is given by:

$$M_{(n:n)}(t) = E\left[e^{ty}\ g_{(n:n)} \right]$$

$$= \int_0^\infty \frac{2}{\sigma^2} e^{ty}\ y\, e^{-y^2/\sigma^2} dy + \int_0^\infty \frac{2n}{\sigma^2 B(2,n)} e^{ty}\ y\ e^{-y^2/2\sigma^2} dy - \frac{4n}{\sigma^2 B(2,n)} \int_0^\infty e^{ty} y\, e^{-y^2/\sigma^2} dy \quad \ldots\mathbf{(C)}$$

This expression for moment generating function can be solved as

$$M_{(n:n)}(t) = \frac{2\sqrt{z}\, e^{\sigma t\sqrt{z}-z}}{\sigma t - 2\sqrt{z}} + \frac{2n}{B(2,n)} \frac{2\sqrt{z}\, e^{\sigma t\sqrt{z}-z}}{\sigma t - \sqrt{2z}} - \frac{2n}{B(2,n)} \frac{2\sqrt{z}\, e^{\sigma t\sqrt{z}-z}}{\sigma t - 2\sqrt{z}} \quad \ldots(15)$$

Now to obtain the moments by expansion method, one can use the following expression.

$$\mu^1_r = Coefficient\ of\ \frac{t^r}{\lfloor r}$$

Differentiating is this manner upto k times we get the k^{th} moment of $y_{(r:n)}$:

$$\frac{d^k}{dt^k} \mu_{(r:n)}(t) = \frac{2\sqrt{z}\, e^{\sigma t\sqrt{z}-z}}{\sigma t - 2\sqrt{z}} + \frac{2n}{B(2,n)} \frac{\sqrt{2z}\, e^{\sigma t\sqrt{2z}-z}}{\sigma t - \sqrt{2z}} - \frac{2n}{B(2,n)} \frac{2\sqrt{z}\, e^{\sigma t\sqrt{2z}-z}}{\sigma t - 2\sqrt{z}}$$

$$\mu_1 = \frac{2\sqrt{z}\, e^{\sigma t\sqrt{z}-z}}{\sigma t - 2\sqrt{z}} + \frac{2\sqrt{2z}\, e^{\sigma t\sqrt{2z}-z}}{B(2,1)\sigma t - \sqrt{2z}} - \frac{2.2\sqrt{z}\, e^{\sigma t 2\sqrt{z}-z}}{B(2,1)\sigma t - 2\sqrt{z}}$$

$$\mu_2 = \frac{2\sqrt{z}\, e^{\sigma t\sqrt{z}-z}}{\sigma t - 2\sqrt{z}} + \frac{4\sqrt{2z}\, e^{\sigma t\sqrt{2z}-z}}{B(2,2)\sigma t - \sqrt{2z}} - \frac{4\sqrt{z}\, e^{\sigma t 2\sqrt{z}-z}}{B(2,2)\sigma t - 2\sqrt{z}}$$

4. Joint distribution of two concomitants $y_{(r:n)}$ and $y_{(s:n)}$:

The joint p.d.f of $Y_{(r:n)}$ and $Y_{(s:n)}$ is given by

$$g_{(r:s:n)}(y_1, y_2) = \int_0^{\infty}\int_0^{x_2} f(y_1/x_1)\, f(y_2/x_2)\, f_{r:s:n}(x_1\, x_2)\, dx_1\, dx_2 \qquad \text{….(16)}$$

The joint p.d.f of $X_{r:n}$ and $X_{s:n}$ for the bivariate Rayleigh distribution with pdf (2).

$$f_{(r:s:n)}(x_1, x_2) = C_{r:s:n}\left[F(x_1)\right]^{r-1}\left[F(x_2)-F(x_1)\right]^{s-r-1}\left[1-f(x_2)\right]^{n-s} f(x_1)\, f(x_2) \qquad \text{….(17)}$$

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$

$$= C_{r:s:n}\sum_{j=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{r-1-j}(-1)^{s-r-1-k}\, r-1_{C_j}\; s-r-1_{C_k}$$

$$\left[\exp\left(\frac{-x_1^2}{2\sigma^2}\right)\right]^{r-j+k}\left[\exp\left(\frac{-x_2^2}{2\sigma^2}\right)\right]^{n-r-k}\frac{x_1 x_2}{\sigma^4} \qquad \text{…(18)}$$

Putting the value eq.(18) in eq.(16)

$$g_{(r:s:n)}(y_1, y_2) = \int_0^{\infty}\int_0^{x_2} f(y_1/x_1)\, f(y_2/x_2)\, f_{r:s:n}(x_1\, x_2)\, dx_1\, dx_2$$

$$= \frac{4y_1y_2}{\sigma^8}C_{r:s:n}\sum_{j=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{r-1-j}(-1)^{s-r-1-k}\, r-1_{C_j}\; s-r-1_{C_k}$$

$$\int_0^{x_2}\left[\exp\left(\frac{-y_1^2}{2\sigma^2}\right)+\exp\left[-\left(\frac{x_1^2+y_1^2}{2\sigma^2}\right)\right]-2\exp\left[-\left(\frac{x_1^2+2y_1^2}{2\sigma^2}\right)\right]\right]\left[\exp\left(-\frac{x_1^2}{2\sigma^2}\right)\right]^{r-j+k} x_1\, dx_1$$

$$\int_0^{\infty}\left[\exp\left(\frac{-y_2^2}{\sigma^2}\right)+\exp\left[-\left(\frac{x_2^2+y_2^2}{2\sigma^2}\right)\right]-2\exp\left[-\left(\frac{x_2^2+2y_2^2}{2\sigma^2}\right)\right]\right]\left[\exp\left(-\frac{x_2^2}{2\sigma^2}\right)\right]^{n-r+k} x_2\, dx_2 \qquad \text{…(D)}$$

This expression is solved as

$$= \frac{4y_1\, y_2}{\sigma^4}C_{r:s:n}\sum_{j=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{r-1-j}(-1)^{s-r-1-k}\, r-1_{C_j}\; s-r-1_{C_k}$$

$$\left[\exp\left(\frac{-y_1^2}{\sigma^2}\right)\left(\frac{\exp\left(-(r-j+k)\frac{x_2^2}{2\sigma^2}-1\right.}{-(r-j+k)}\right)+\exp\left(\frac{-y_1^2}{2\sigma^2}\right)\right.$$

$$\frac{\exp\left(-(r-j+k+1)\frac{x_2^2}{2\sigma^2}-1\right.}{-(r-j+k+1)}-2\exp\left(\frac{-y_1^2}{\sigma^2}\right)$$

$$\left[\frac{\exp\left(-(r-j+k+1)\frac{x_2^2}{2\sigma^2}-1\right.}{-(r-j+k+1)}\right]\left[-\frac{\exp\left(\frac{-y_2^2}{\sigma^2}\right)}{(n-r-k)}-\frac{\exp\left(\frac{-y_2^2}{2\sigma^2}\right)}{(n-r-k+1)}+\frac{2\exp\left(\frac{-y_2^2}{\sigma^2}\right)}{(n-r-k+1)}\right] \qquad \text{…(19)}$$

Conclusion

In this chapter a new bivariate distribution named as Rayleigh Bivaraite distribution by using lines of Mongestern (1956) is developed. The distribution of order statistics and their induced statistics had been obtained. To characterize the distribution the moments of concomitants of order statistics have been also obtained.

References

1. Balasubramnian K, M. I. Beg. Concomitants of Order Statistics in Morgenstern Type Bivariate Exponential Distribution. *J. App. Statist. Sci. 1997;54(4):233-245.*
2. Bhattacharya, P.K. Induced Order Statistics: Theory and Applications. In: Krishnaiah, P.R and Sen, P.K. (Eds.). Hand Book Of Statistics 4, 1984;383-403, *Elsevier Science.*
3. David H.A. (1982) : "Concomitants Of Order Statistics: Theory & Applications", In Some Recent Advances In Statistics, 89-100, *Academic Press, NY.*
4. Johnson, N.L. And Kotz, S. (1970): "Distribution In Statistics, Continuous Univariate Distributions-2", *John Wiley And Sons, NY.*
5. Morgenstern D. Einfache beispiele zweidimensionaler verteilungen. Mitteilungsblatt fürMathematische Statistik 1956;8:234–5.